Back Paper ISI Bangalore June 2019

Introduction to Algebraic Geometry 100 Points

Notes.

(a) Justify all your steps. You may use any result proved in class unless you have been asked to prove the same.

(b) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.

(c) By default, k denotes an algebraically closed field and $\mathbb{A}_k^n$ is the affine n-space over k while $\mathbb{P}_k^n$ is the projective n-space over k. By default, the polynomial ring of functions on $\mathbb{A}_k^n$ is denoted as $k[x_1, \ldots, x_n]$ while for $n = 1, 2, 3$ we also use the usual notation of x, y, z for the variables.

(d) We will use $\mathcal{V}(-)$ to denote the common zero locus (in suitable affine or projective space) of any collection of polynomials and $\mathcal{I}(-)$ the ideal of functions vanishing on a given subset of affine or projective space.

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1. [20 points] Let X, Y be affine algebraic sets over k. Show that there is a one-one correspondence between the set of all polynomial maps $X \to Y$ and the set of all k-algebra homomorphisms $k[Y] \to k[X]$ where $k[X], k[Y]$ denote the coordinate rings of X, Y respectively.

2. [20 points] What is a projective line in $\mathbb{P}^2$? Prove that any two (projective) lines in $\mathbb{P}^2$ intersect. Find two lines in $\mathbb{P}^3$ that do not intersect.

3. [20 points] For any irreducible quasi-projective variety W, define the field of rational functions $K(W)$ on W. Prove that if W is the projective variety given by $y^2z - x^3 = 0$ in $\mathbb{P}^2$ then the field of rational functions on W is isomorphic to $k(T)$, the fraction field of the polynomial ring $k[T]$.

4. [20 points] Prove that any noetherian topological space X can be uniquely written as a finite union of irreducible closed subsets X_i where X_i is not contained in the union of the remaining subsets X_j. Prove that these X_i are maximal among irreducible subsets, that is, if $X_i \subset Z$ where $Z \subset X$ is irreducible, then $X_i = Z$. Find such a decomposition into irreducibles for the affine variety $\mathcal{V}(xz, yz)$ in $\mathbb{A}^3$.

5. [20 points] Define the Krull dimension of a noetherian topological space. Calculate the Krull dimension of the following algebraic sets.

    (i) $\mathcal{V}(z(x, z-1)(x+1, y, z+1)) \subset \mathbb{A}^3$.

    (ii) $\mathcal{V}(xz - y^2, yw - z^2, xw - yz) \subset \mathbb{P}^3$.